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# On multidimensional inverse scattering in time-dependent electric fields<sup>†</sup>

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## 1 Introduction

Throughout this paper, we assume the spatial dimension  $d \geq 2$ . We report one of the inverse scattering problems for quantum systems in a time-dependent electric field  $E(t) \in \mathbb{R}^d$ , which was obtained in Adachi-Fujiwara-I [1]. By Enss-Weder time-dependent method [5], we can show that the high speed limit of the scattering operator determines uniquely the potential  $V$  belonging to the wider class than the classes given by the previous work in Adachi-Maehara [3], Adachi-Kamada-Kazuno-Toratani [2], Nicoleau [10] and Fujiwara [6].

The free and full Hamiltonians under the consideration are given by

$$H_0(t) = p^2/2 - E(t) \cdot x, \quad H(t) = H_0(t) + V \quad (1.1)$$

acting as the self-adjoint operators on  $L^2(\mathbb{R}^d)$ , where  $p = -i\nabla_x$  is the momentum,  $E(t)$  is the time-dependent electric field and the interaction potential  $V$  is real-valued multiplicative operator.  $E(t)$  and  $V = V^{\text{vs}} + V^{\text{s}} + V^{\text{l}} \in \mathcal{V}^{\text{vs}} + \mathcal{V}_{\mu, \alpha_\mu}^{\text{s}} + \mathcal{V}_{\mu, \gamma_\mu}^{\text{l}}$  satisfy following assumptions.

**Assumption 1.1.** *The time-dependent electric field  $E(t) \in \mathbb{R}^d$  is represented as*

$$E(t) = E_0(1 + |t|)^{-\mu} + E_1(t), \quad (1.2)$$

where  $0 \leq \mu < 1$ ,  $E_0 \in \mathbb{R}^d \setminus \{0\}$  and  $E_1(t) \in C(\mathbb{R}, \mathbb{R}^d)$  such that

$$\left| \int_0^t \int_0^s E_1(\tau) d\tau ds \right| \leq C \max\{|t|, |t|^{2-\mu_1}\} \quad (1.3)$$

with  $\mu < \mu_1 \leq 1$ .

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<sup>†</sup>Joint work with Tadayoshi ADACHI (Kobe University) and Yuko FUJIWARA

Roughly speaking about the perturbation part  $E_1(t)$ , we assume that  $|E_1(t)| \leq C(1 + |t|)^{-\mu_2}$  for some  $\mu_2 > \mu$  and take  $\mu_1$  as follows:

$$\begin{cases} \mu_1 = \mu_2 & \mu < \mu_2 < 1 \\ \mu < \mu_1 < \mu_2 & \mu_2 = 1 \\ \mu_1 = 1 & \mu_2 > 1. \end{cases} \quad (1.4)$$

Such  $E(t)$  was first dealt with in Adachi-Kamada-Kazuno-Toratani [2]. For brevity's sake, we suppose that  $E_0 = e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$ .

**Assumption 1.2.**  $\mathcal{V}^{\text{vs}}$  is the class of real-valued multiplicative operators  $V^{\text{vs}}$  is satisfying that  $V^{\text{vs}}$  is decomposed into the sum of a singular part  $V_1^{\text{vs}}$  and a regular part  $V_2^{\text{vs}}$ .  $V_1^{\text{vs}}$  is compactly supported, belongs to  $L^{q_1}(\mathbb{R}^d)$  and satisfies  $|\nabla V_1^{\text{vs}}| \in L^{q_2}(\mathbb{R}^d)$ .  $V_2^{\text{vs}} \in C^1(\mathbb{R}^d)$  satisfies that  $V_2^{\text{vs}}$  and its first derivatives are all bounded in  $\mathbb{R}^d$  and that

$$\int_0^\infty \|F(|x| \geq R)V_2^{\text{vs}}(x)\|_{\mathcal{B}(L^2)} dR < \infty. \quad (1.5)$$

Here  $q_1$  satisfies that  $q_1 > d/2$  and  $q_1 \geq 2$ ,  $q_2$  satisfies

$$\begin{cases} 1/q_2 = 1/(2q_1) + 2/d & d \geq 5 \\ 1/q_2 < 1/(2q_1) + 1/2 & d = 4 \\ 1/q_2 = 1/(2q_1) + 1/2 & d \leq 3, \end{cases} \quad (1.6)$$

and  $F(|x| \geq R)$  is the characteristic function of  $\{x \in \mathbb{R}^d \mid |x| \geq R\}$ .

$\mathcal{V}_{\mu, \alpha_\mu}^s$  with some  $\alpha_\mu > 0$  is the class of real-valued multiplicative operators  $V^s$  is satisfying that  $V^s$  belongs to  $C^1(\mathbb{R}^d)$  and satisfies

$$|V^s(x)| \leq C\langle x \rangle^{-\gamma}, \quad |\partial_x^\beta V^s(x)| \leq C_\beta \langle x \rangle^{-1-\alpha}, \quad |\beta| = 1 \quad (1.7)$$

with some  $\gamma$  and  $\alpha$  such that  $1/(2-\mu) < \gamma \leq 1$  and  $\alpha_\mu < \alpha \leq \gamma$ .

Finally,  $\mathcal{V}_{\mu, \gamma_\mu}^1$  with some  $\gamma_\mu \geq 1/(2(2-\mu))$  is the class of real-valued multiplicative operators  $V^1$  is satisfying that  $V^1$  belongs to  $C^2(\mathbb{R}^d)$  and satisfies

$$|\partial_x^\beta V^1(x)| \leq C\langle x \rangle^{-\gamma_D - |\beta|/(2-\mu)}, \quad |\beta| \leq 2, \quad (1.8)$$

with some  $\gamma_D$  such that  $\gamma_\mu < \gamma_D \leq 1/(2-\mu)$ .

We note that one can obtain

$$\int_0^\infty \|F(|x| \geq R)V^{\text{vs}}(x)\langle p \rangle^{-2}\|_{\mathcal{B}(L^2)} dR < \infty \quad (1.9)$$

by this assumption and it is equivalent to

$$\int_0^\infty \|V^{\text{vs}}(x)\langle p \rangle^{-2}F(|x| \geq R)\|_{\mathcal{B}(L^2)} dR < \infty \quad (1.10)$$

because  $V^{vs}$  is a multiplicative operator (see e.g. Reed-Simon [11]).

As for the class  $\mathcal{V}_{\mu, \alpha_\mu}^s$ , we also note that by virtue of  $\alpha \leq \gamma$ , we can treat an oscillation part. For example, the following function belongs to  $\mathcal{V}_{\mu, \alpha_\mu}^s$ :

$$V^s(x) = \langle x \rangle^{-\gamma} \cos \langle x \rangle^{\gamma-\alpha}. \quad (1.11)$$

In fact, we can verify easily that  $|\nabla_x V^s(x)| \leq C(\langle x \rangle^{-1-\gamma} + \langle x \rangle^{-1-\alpha}) \leq C\langle x \rangle^{-1-\alpha}$  holds with some  $C > 0$ .

## 2 Results

We first state the case where  $V^1 = 0$ . Then we can see the wave operators

$$W^\pm = \text{s-lim}_{t \rightarrow \pm\infty} U(t, 0)^* U_0(t, 0) \quad (2.1)$$

exist as this fact was shown in Adachi-Kamada-Kazuno-Toratani [2], where we denote the propagators generated by  $H_0(t)$  and  $H(t)$  as  $U_0(t, 0)$  and  $U(t, 0)$ . The existence and uniqueness of these propagators are guaranteed by virtue of Yajima [14]. The scattering operator  $S = S(V)$  is defined by

$$S = (W^+)^* W^-. \quad (2.2)$$

The following obtained in [1] is one of those which we would like to report in this paper.

**Theorem 2.1. (Adachi-Fujiwara-I [1])** *Put*

$$\tilde{\alpha}_\mu = \begin{cases} \frac{7 - 3\mu - \sqrt{(1-\mu)(17-9\mu)}}{4(2-\mu)} & 0 \leq \mu \leq 1/2 \\ \frac{1+\mu}{2(2-\mu)} & 1/2 < \mu < 1. \end{cases} \quad (2.3)$$

*Let  $V_1, V_2 \in \mathcal{V}^{vs} + \mathcal{V}_{\mu, \tilde{\alpha}_\mu}^s$ . If  $S(V_1) = S(V_2)$ , then  $V_1 = V_2$ .*

In the case where  $E(t) \equiv E_0$ , that is, the case of the Stark effect, this theorem was first proved by Weder [12] under the condition  $V^s \in \mathcal{V}_{0,0}^s$  and the additional assumption  $\gamma > 3/4$ . However, as it is well-known, the short-range condition on  $V$  under the Stark effect is  $\gamma > 1/2$ . Later Nicoleau [9] proved this theorem for real-valued  $V \in C^\infty(\mathbb{R}^d)$  satisfying  $|\partial_x^\beta V(x)| \leq C_\beta \langle x \rangle^{-\gamma-|\beta|}$  with  $\gamma > 1/2$ , under the spatial dimension  $d \geq 3$ . After that, this theorem was obtained by Adachi-Maehara [3] for  $V^s \in \mathcal{V}_{0,1/2}^s$ . In our case where  $\mu = 0$ , substitute  $\mu = 0$  in  $\tilde{\alpha}_\mu$ . We have

$$\tilde{\alpha}_0 = \left. \frac{7 - 3\mu - \sqrt{(1-\mu)(17-9\mu)}}{4(2-\mu)} \right|_{\mu=0} = \frac{7 - \sqrt{17}}{8} < \frac{1}{2}. \quad (2.4)$$

If  $a < b$ , then  $\mathcal{V}_{\mu,b}^s \subsetneq \mathcal{V}_{\mu,a}^s$ . Therefore this implies

$$\mathcal{V}_{0,1/2}^s \subsetneq \mathcal{V}_{0,\tilde{\alpha}_0}^s. \quad (2.5)$$

In the time-dependent case where  $0 < \mu < 1$  and  $E_1(t) \not\equiv 0$ , the result corresponding to Theorem 2.1 was also obtained by Adachi-Kamada-Kazuno-Toratani [2] under the assumption that  $V \in \mathcal{V}^{\text{vs}} + \mathcal{V}_{\mu, 1/(2-\mu)}^{\text{s}}$ . We can verify that  $\tilde{\alpha}_\mu < 1/(2-\mu)$  and this implies that above result is finer than the previous one:

$$\mathcal{V}_{\mu, 1/(2-\mu)}^{\text{s}} \subsetneq \mathcal{V}_{\mu, \tilde{\alpha}_\mu}^{\text{s}}. \quad (2.6)$$

We next state the case where  $V^1 \neq 0$ . If  $V^1 \in \mathcal{V}_{\mu, \gamma_\mu}^1$ , the Dollard-type modified wave operators due to White [13] (see also Adachi-Tamura [4] and Jensen-Yajima [8])

$$W_D^\pm = \text{s-lim}_{t \rightarrow \pm\infty} U(t, 0)^* U_0(t, 0) M_D(t), \quad M_D(t) = e^{-i \int_0^t V^1(p\tau + c(\tau)) d\tau} \quad (2.7)$$

can exist by virtue of the condition  $\gamma_D > 1/(2(2-\mu))$  (see [2]), where we put

$$c(t) = \int_0^t b(\tau) d\tau, \quad b(t) = \int_0^t E(\tau) d\tau. \quad (2.8)$$

Then the Dollard-type modified scattering operator  $S_D = S_D(V^1, V^{\text{vs}} + V^{\text{s}})$  is defined by

$$S_D = (W_D^+)^* W_D^-. \quad (2.9)$$

Then we also report the following result.

**Theorem 2.2. (Adachi-Fujiwara-I [1])** *Suppose that a given  $V^1$  satisfies  $V^1 \in \mathcal{V}_{\mu, \tilde{\gamma}_\mu}^1$  with*

$$\tilde{\gamma}_\mu = \frac{1}{2(2-\mu)} + \frac{1-\mu}{4(2-\mu)}. \quad (2.10)$$

*Put*

$$\tilde{\alpha}_{\mu, D} = \begin{cases} \frac{13 - 5\mu - \sqrt{(1-\mu)(41-25\mu)}}{8(2-\mu)} & 0 \leq \mu \leq 5/7 \\ \frac{1+\mu}{2(2-\mu)} & 5/7 < \mu < 1. \end{cases} \quad (2.11)$$

*Let  $V_1, V_2 \in \mathcal{V}^{\text{vs}} + \mathcal{V}_{\mu, \tilde{\alpha}_{\mu, D}}^{\text{s}}$ . If  $S_D(V^1, V_1) = S_D(V^1, V_2)$ , then  $V_1 = V_2$ . Moreover, any one of the Dollard-type modified scattering operators  $S_D$  determines uniquely the total potential  $V$ .*

In the case where  $0 < \mu < 1$  and  $E_1(t) \not\equiv 0$ , Adachi-Kamada-Kazuno-Toratani [2] proved this theorem under the condition that

$$V^1 \in \mathcal{V}_{\mu, \hat{\gamma}_\mu}^1, \quad V_1, V_2 \in \mathcal{V}^{\text{vs}} + \mathcal{V}_{\mu, 1/(2-\mu)}^{\text{s}} \quad (2.12)$$

with  $\hat{\mu} = (7 - \sqrt{3} - \sqrt{60 - 22\sqrt{3}})/4$  and

$$\hat{\gamma}_\mu = \begin{cases} \frac{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(1-\mu)^2}{1+\mu}}}{2-\mu} & 0 < \mu \leq \hat{\mu} \\ -\frac{3-\mu}{8} + \sqrt{\frac{(3-\mu)^2}{64} + \frac{2\mu^2 - 7\mu + 7}{4(2-\mu)^2}} & \hat{\mu} < \mu < 1. \end{cases} \quad (2.13)$$

Computing straightforwardly, we can see  $\tilde{\alpha}_{\mu,D} < 1/(2-\mu)$  and  $\tilde{\gamma}_\mu < \hat{\gamma}_\mu$ . We thus obtain

$$\mathcal{V}_{\mu,1/(2-\mu)}^s \subsetneq \mathcal{V}_{\mu,\tilde{\alpha}_{\mu,D}}^s, \quad \mathcal{V}_{\mu,\tilde{\gamma}_\mu}^1 \subsetneq \mathcal{V}_{\mu,\hat{\gamma}_\mu}^1. \quad (2.14)$$

In particular, there was no result for the case where  $\mu = 0$ . Here we emphasize that if  $5/7 \leq \mu < 1$ , then  $\tilde{\alpha}_{\mu,D} = \tilde{\alpha}_\mu$  holds, although if  $0 \leq \mu < 5/7$ , then  $\tilde{\alpha}_{\mu,D} > \tilde{\alpha}_\mu$  holds.

**Remark 2.3.** We assume that  $E(t) \in C(\mathbb{R}, \mathbb{R}^d)$  is  $T$ -periodic in time with non-zero mean  $E_0$ , that is,

$$E_0 = \int_0^T E(\tau) d\tau / T \neq 0, \quad (2.15)$$

which was treated by Nicoleau [10] and Fujiwara [6]. In this case, the method in the proofs of Theorems 2.1 and 2.2 does work well also, because we have

$$|b(t) - tE_0| \leq \int_0^T |E(\tau) - E_0| d\tau, \quad (2.16)$$

$$|c(t) - t^2 E_0 / 2| \leq \int_0^{|t|} |b(\tau) - \tau E_0| d\tau \leq C|t|, \quad (2.17)$$

with  $C = \int_0^T |E(\tau) - E_0| d\tau$  by the periodicity of  $E(t)$ . (2.17) implies  $\mu = 0$  in (1.2) and  $\mu_1 = 1$  in (1.3).

By virtue of this fact, we can obtain an improvement of the results of [10] and [6].

**Theorem 2.4.** Suppose that  $E(t) \in C(\mathbb{R}, \mathbb{R}^d)$  is  $T$ -periodic in time with non-zero mean  $E_0$ . Then the followings hold.

1. Let  $V_1, V_2 \in \mathcal{V}^{\text{vs}} + \mathcal{V}_{0,\tilde{\alpha}_0}^s$ . If  $S(V_1) = S(V_2)$ , then  $V_1 = V_2$ .
2. Suppose that a given  $V^1$  satisfies  $V^1 \in \mathcal{V}_{0,\tilde{\gamma}_0}^1$ . Let  $V_1, V_2 \in \mathcal{V}^{\text{vs}} + \mathcal{V}_{0,\tilde{\alpha}_{0,D}}^s$ . If  $S_D(V^1, V_1) = S_D(V^1, V_2)$ , then  $V_1 = V_2$ . Moreover, any one of the Dollard-type modified scattering operators  $S_D$  determines uniquely the total potential  $V$ .

Nicoleau [10] proved the uniqueness assuming that  $|\partial_x^\beta V(x)| \leq C_\beta \langle x \rangle^{-\gamma-|\beta|}$  with  $\gamma > 1/2$  for  $V \in C^\infty(\mathbb{R})^d$  and the additional condition  $d \geq 3$ . Fujiwara [6] assumed that  $V \in \mathcal{V}^{\text{vs}} + \mathcal{V}_{0,1/2}^s$ . These two results did not treat the long-range potentials.

### 3 Short-range Case

By virtue of Theorem 3.1 below and the Plancherel formula associated with the Radon transform (see Helgason [7]), Theorem 2.1 can be shown in the quite same way as in the proof of Theorem 1.2 in [12] (see also Enss-Weder [5]).

**Theorem 3.1. (Reconstruction Formula [1])** Let  $\hat{v} \in \mathbb{R}^d$  be given such that  $|\hat{v} \cdot e_1| < 1$ . Put  $v = |v|\hat{v}$ . Let  $\eta > 0$  be given, and  $\Phi_0, \Psi_0 \in \mathcal{S}(\mathbb{R}^d)$  be such that  $\mathcal{F}\Phi_0, \mathcal{F}\Psi_0 \in C_0^\infty(\mathbb{R}^d)$  with  $\text{supp } \mathcal{F}\Phi_0, \text{supp } \mathcal{F}\Psi_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq \eta\}$ . Put  $\Phi_v = e^{iv \cdot x} \Phi_0, \Psi_v = e^{iv \cdot x} \Psi_0$ . Let  $V^{vs} \in \mathcal{V}^{vs}$  and  $V^s \in \mathcal{V}_{\mu, \tilde{\alpha}_\mu}^s$ , where  $\tilde{\alpha}_\mu$  is the same as in Theorem 2.1 and  $\mathcal{F}$  is the Fourier transformation. Then

$$|v|(i[S, p_j]\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} ((V^{vs}(x + \hat{v}t)p_j\Phi_0, \Psi_0) - (V^{vs}(x + \hat{v}t)\Phi_0, p_j\Psi_0) + (i(\partial_{x_j} V^s)(x + \hat{v}t)\Phi_0, \Psi_0))dt + o(1) \quad (3.1)$$

holds as  $|v| \rightarrow \infty$  for  $1 \leq j \leq d$ .

To prove Theorem 3.1, the following estimate is the key.

**Proposition 3.2.** Let  $v$  and  $\Phi_v$  be as in Theorem 3.1 and  $\epsilon > 0$ . Put

$$\Theta(\alpha) = \begin{cases} \alpha + \frac{(\alpha - \mu)(1 - \alpha)}{(1 - \mu)(2 - \alpha)} & \alpha > \mu \\ \alpha - \frac{\mu - \alpha}{1 - \mu} & \mu/(2 - \mu) < \alpha \leq \mu. \end{cases} \quad (3.2)$$

Then

$$\int_{-\infty}^{\infty} \|(V^s(x) - V^s(vt + c(t)))U_0(t, 0)\Phi_v\|dt = O(|v|^{-\Theta(\alpha)+\epsilon}) \quad (3.3)$$

holds as  $|v| \rightarrow \infty$  for  $V^s \in \mathcal{V}_{\mu, \mu/(2-\mu)}^s$ .

In Adachi-Maehara [3], the corresponding estimate to this proposition was

$$\int_{-\infty}^{\infty} \|(V^s(x) - V^s(vt + c(t)))U_0(t, 0)\Phi_v\|dt = O(|v|^{-\alpha}) \quad (3.4)$$

(see Lemma 2.2 in [3]). When we denote the error term of (3.1) by  $R(v)$ ,  $\lim_{|v| \rightarrow \infty} R(v) = 0$  is equivalent to  $2(-\alpha) + 1 < 0$ . Therefore  $\alpha > 1/2$  was required. On the other hand, in Adachi-Kamada-Kazuno-Toratani [2], the corresponding one was

$$\int_{-\infty}^{\infty} \|(V^s(x) - V^s(vt + c(t)))U_0(t, 0)\Phi_v\|dt = O(|v|^{-\rho}), \quad (3.5)$$

where

$$\rho = \frac{(2 - \mu)\alpha - \mu}{2(1 - \mu)} \quad (3.6)$$

(see Lemma 3.4 in [2]) and  $\lim_{|v| \rightarrow \infty} R(v) = 0$  is equivalent to  $2(-\rho) + 1 < 0$ . Solving this inequality for  $\alpha$ , we see that  $\alpha > 1/(2 - \mu)$  was required. In our estimate,  $\alpha > \tilde{\alpha}_\mu$  comes from the inequality  $2(-\Theta(\alpha)) + 1 < 0$  which is equivalent to  $\lim_{|v| \rightarrow \infty} R(v) = 0$ .

## 4 Long-range Case

In the case where  $V^1 \neq 0$ , the reconstruction formula is represented as follows, which also yields the proof of Theorem 2.2.

**Theorem 4.1. (Reconstruction Formula [1])** *Let  $\hat{v} \in \mathbb{R}^d$  be given such that  $|\hat{v} \cdot e_1| < 1$ . Put  $v = |v|\hat{v}$ . Let  $\eta > 0$  be given, and  $\Phi_0, \Psi_0 \in \mathcal{S}(\mathbb{R}^d)$  be such that  $\mathcal{F}\Phi_0, \mathcal{F}\Psi_0 \in C_0^\infty(\mathbb{R}^d)$  with  $\text{supp } \mathcal{F}\Phi_0, \text{supp } \mathcal{F}\Psi_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq \eta\}$ . Put  $\Phi_v = e^{iv \cdot x} \Phi_0, \Psi_v = e^{iv \cdot x} \Psi_0$ . Let  $V^{\text{vs}} \in \mathcal{V}^{\text{vs}}$ ,  $V^s \in \mathcal{V}_{\mu, \tilde{\alpha}_{\mu, D}}^s$  and  $V^1 \in \mathcal{V}_{\mu, \tilde{\gamma}_\mu}^1$ , where  $\tilde{\alpha}_{\mu, D}$  and  $\tilde{\gamma}_\mu$  are the same as in Theorem 2.2. Then*

$$\begin{aligned} |v|(i[S_D, p_j]\Phi_v, \Psi_v) &= \int_{-\infty}^{\infty} ((V^{\text{vs}}(x + \hat{v}t)p_j\Phi_0, \Psi_0) - (V^{\text{vs}}(x + \hat{v}t)\Phi_0, p_j\Psi_0) \\ &\quad + (i(\partial_{x_j}V^s)(x + \hat{v}t)\Phi_0, \Psi_0) + (i(\partial_{x_j}V^1)(x + \hat{v}t)\Phi_0, \Psi_0))dt + o(1) \end{aligned} \quad (4.1)$$

holds as  $|v| \rightarrow \infty$  for  $1 \leq j \leq d$ .

To prove Theorem 4.1, the following estimate is the key.

**Proposition 4.2.** *Let  $v$  and  $\Phi_v$  be as in Theorem 4.1,  $\epsilon > 0$  and  $V^1 \in \mathcal{V}_{\mu, 1/(2(2-\mu))}^1$ . Put*

$$\Theta_D(\gamma_D) = \begin{cases} 1 & \gamma_D > 1/2 \\ \frac{2\gamma_D(2-\mu) - 1}{1-\mu} & \gamma_D \leq 1/2. \end{cases} \quad (4.2)$$

Then

$$\int_{-\infty}^{\infty} \|(V^1(x) - V^1(t(p - b(t)) + c(t)))U_D(t)\Phi_v\|dt = O(|v|^{-\Theta_D(\gamma_D)+\epsilon}) \quad (4.3)$$

holds as  $|v| \rightarrow \infty$ , where  $U_D(t) = U_0(t, 0)M_D(t)$  and  $M_D(t)$  is the same as in (2.7).

In Adachi-Kamada-Kazuno-Toratani [2], the corresponding estimate to this proposition was

$$\int_{-\infty}^{\infty} \|(V^1(x) - V^1(t(p - b(t)) + c(t)))U_D(t)\Phi_v\|dt = O(|v|^{-\rho_1}), \quad (4.4)$$

where

$$\rho_1 = \frac{(1 - \sigma_\kappa)(\tilde{\gamma}_D + 2 - \mu)}{(1 - \mu)(\sigma_\kappa(\tilde{\gamma}_D + 2 - \mu) - 1 + \tilde{\gamma}_D)} \quad (4.5)$$

with  $\tilde{\gamma}_D = (2 - \mu)\gamma_D$  and  $\sigma_\kappa = 1 - \kappa(1 - \mu)/(2 - \mu)$  for  $0 < \kappa < 1$  (see Lema 4.5 in [2]). When we denote the error term of (4.1) by  $R_D(v)$ ,  $\lim_{|v| \rightarrow \infty} R_D(v) = 0$  is equivalent to  $2(-\rho_1) + 1 < 0$ . Therefore  $\gamma_D > \hat{\gamma}_\mu$  was required. In our case,  $\lim_{|v| \rightarrow \infty} R_D(v) = 0$  is equivalent to  $2(-\Theta_D(\gamma_D)) + 1 < 0$  and  $\gamma_D > \hat{\gamma}_\mu$  comes from this inequality.



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